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M.A./M.Sc. (THIRD SEMESTER) EXAMINATION, Dec. - Jan., 2021-22

MATHEMATICS

(PAPER FIRST)

(Integration Theory and Functional Analysis - I)

Time : Three Hours] ***[Maximum Marks : 80***

[Minimum Pass Marks : 16

Note : Attempt all sections as directed.

Section - A

(1 mark each)

(Objective /Multiple Choice Questions)

Note : Attempt all questions:

Choose the correct answer:

1. A union of any countable collection of positive subset of X is
   - (A) Positive
   - (B) Negative
   - (C) Both positive and Negative
   - (D) None of the above

2. Two signed measure μ and γ for which both $\gamma << \mu$ and $\mu << \gamma$ are called
   - (A) Singular
   - (B) Absolute continuous
   - (C) Equivalent
   - (D) None of the above

3. If μ is a signed measure and f is a measurable function such that f is integrable with respect to $|\mu|$ then
   - (A) $\int f\,d\mu = \int f\,d\mu^{+} + \int f\,d\mu^{-}$
   - (B) $\int f\,d\mu = \int f\,d\mu^{+} - \int f\,d\mu^{-}$
   - (C) $\int f\,d\mu \neq \int f\,d\mu^{+} - \int f\,d\mu^{-}$
   - (D) None of the above

4. The collection A^* of all μ^* - measurable set is σ- algebra containing A & *if* $< E_n >$ is a disjoint sequence in A^* then.
   - (A) $\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) > \sum_{n=1}^{\infty} \mu^*(E_n)$
   - (B) $\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) < \sum_{n=1}^{\infty} \mu^*(E_n)$
   - (C) $\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu^*(E_n)$
   - (D) $\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) \neq \sum_{n=1}^{\infty} \mu^*(E_n)$

5. An absolutely continuous function is _______ almost every where.
   (A) Continuous
   (B) Differentiable
   (C) Increasing
   (D) None of the above
6. If a function f is absolutely continuous in an open interval (a, b) and if $f'(x) = 0$ almost every where in [a, b] then f is.
   (a) A finite value
   (B) Constant
   (C) Both (A) and (B)
   (D) None of these
7. If the derivative of two absolutely continuous function are equivalent then the
   (A) Function differ by a constant
   (B) Function differ by a variable
   (C) Function is not constant
   (D) None of these
8. If X is the characteristic function of E of $X \times Y$ then X_x and x^y are the characteristic function of a rectangle $A \times B$ then.
   (A) $X^{(x,y)} = X_A(x) + X_B(y)$
   (B) $X^{(x,y)} = X_A(x) - X_B(y)$
   (C) $X^{(x,y)} = X_A(x).X_B(y)$
   (D) None of these

9. Every compact Baire set is
   (A) G_δ
   (B) F_σ
   (C) Both (A) and (B)
   (D) None of these
10. If X is any topological space, the smallest σ-algebra containing the closed set is called
   (A) Baire set
   (B) Borel set
   (C) Locally compact
   (D) None of these
11. A finite disjoint union of inner regular set of finite measure is-
   (A) Outer Regular
   (B) Regular
   (C) Inner Regular
   (D) None of these
12. Let μ be a Baire measure such that the measure of every non-empty Baire open set is positive and if $f \in J_+$ then a necessary and sufficient condition that $\int f \, d\mu = 0$ is that
   (A) $f(x) = 0 \;\; \forall \; x \in X$
   (B) $f(x) > 0 \;\; \forall \; x \in X$
   (C) $f(x) < 0 \;\; \forall \; x \in X$
   (D) $f(x) \neq 0 \;\; \forall \; x \in X$

13. Every complete subspace of a normed linear space is.

(A) Open

(B) Closed

(C) Both open and closed

(D) None of these

14. A normed linear space *X* is complete if and only if every absolutely convergent series in *X* is.

(A) Complete

(B) Sequentially complete

(C) Convergent

(D) None of these

15. Two normed linear space X and Y are said to _______ if and only if they are isometrically isomorphic

(A) Isomorphic

(B) Homeomorphic

(C) Finite

(D) Equivalent

16. Any finite dimensional normed linear space is.

(A) Banach space

(B) Linear space

(C) Metric space

(D) None of these

17. Let $1 < P < \infty$ and $\frac{1}{P} + \frac{1}{q} = 1$ then.

(A) dual of ℓ_p is ℓ_q*

(B) dual of ℓ_p is ℓ_q

(C) dual of ℓ_p is ℓ_∞

(D) None of these

18. Let *X* and *Y* be normed linear space and *T a* linear transformation on *X* into *Y* then *T* is continuous if and only if there is a constant M such that

(A) $\|Tx\| = M\|x\| \ \forall x \in X$

(B) $\|Tx\| \geq M\|x\| \ \forall x \in X$

(C) $\|Tx\| \leq M\|x\| \ \forall x \in X$

(D) None of these

19. If *X* and *Y* and normed linear space and *T* is *a* linear operator on *X* into *Y* then *T* is continuous if and only *if T* is

(A) Uniformly continuous

(B) Bounded

(C) Continuous at one point

(D) None of these

20. Let $\{x_n\}$ be a weakly convergent sequence in a normed space X i.e. $x_n \xrightarrow{w} x$ then.

Statement I : Every subsequence of $\{x_n\}$ converges weakly to x.

Statement II : The sequence $\|x_n\|$ is bounded.

(A) Statement I is true and statement II is false

(B) Statement I is false and statement II is true

(C) Both statements I and II are true

(D) Both statements I and II are false

Section - B

(1½ marks each)

(Very short answer type questions)

Note - Attempt all questions.

1. Define totally finite signed measure.
2. Define Radon Nikodym theorem.
3. Define locally bounded variation.
4. Define absolutely continuous function.
5. Define Regular measure.
6. Define locally compact space.
7. Give an example of normed linear space.
8. State Riesz Lemma.
9. Define weak and strong convergence.
10. Define bounded linear functional.

Section - C

(2½ marks each)

Note - Attempt all questions.

1. Let *E* and *F* be measurable set and μ is a signed measure such that $E \subset F$ and $|\mu(F)| < \infty$ then $|\mu(E)| < \infty$.
2. Let μ and γ are signed measure then the conditions.

   (i) $\gamma << \mu$

   (ii) $\gamma^+ << \mu$ and $\gamma^- << \mu$

   (iii) $|\gamma| << |\mu|$

   are mutually equivalent.
3. Prove that an integral function is a continuous function.
4. Let $\{(Ai \times Bi)\}$ be a countable disjoint collection of measurable rectangle whose union is a measurable rectangle *A* × *B* then

$$\lambda(A \times B) = \sum_i \lambda(A_i \times B_i)$$

5. Every Borel set is σ-bounded if and only if every σ-bounded open set is a Borel set.
6. Prove that a finite disjoint union of inner regular set of finite measure is inner regular.
7. In a normed linear space prove that every convergent sequence is a Cauchy sequence.
8. State and prove Riesz Lemma.

9. Prove that in a finite dimensional normed linear space strong and weak convergence coincide.

10. Let X and Y be normed linear space and T : X$\rightarrow$Y a linear transformation. Then the following three conditions are equivalent.

    (i) T is bounded

    (ii) T is continuous

    (iii) T is continuous at one point

Section - D

(4 marks each)

(Long answer type questions)

Note - Attempt all questions.

1. Let *E* be a measurable set of finite measure is $0 < \mu(E) < \infty$ then *E* contains a positive set A with $\mu(\text{A}) > 0$

OR

State and prove Random Nikodym theorem.

2. State and prove Fubini's theorem

OR

Prove that if a function *f* is absolutely continuous in an open interval (a, b) and if f'(x) = 0 almost every where in [a, b] then *f* is constant.

3. Prove that every compact Baire set is G_δ.

OR

A necessary and sufficient condition that every set in $\hat{\text{C}}$ be outer regular is that every bounded set in $\hat{\text{U}}$ be inner regular.

4. Prove that a normed linear space X is complete if and only if every absolutely convergent series in X is convergent

OR

Let X be a normed linear space. The closed unit ball $\text{B} = \{x \in X : \| x \| < 1\}$ in X is compact if and only if X is finite dimensional.

5. Let X and Y be normed linear space. Then B (X, Y) the set of all bounded linear transformation from X into Y is a normed linear space. More over if Y is a Banach space then B(X, Y) is also a Banach space.